8/2020

B. Sc. Part 2 — 3rd Paper

Group B

Infinite series (contd.)

RAABE'S TEST

$$\lim_{n\to\infty} n\left(\frac{U_n}{U_{n+1}} - 1\right) > 1 \Rightarrow \text{convergent series}$$

$$< 1 \Rightarrow \text{divergent series}$$

$$= 1 \Rightarrow \text{Raabe's test fails}$$

Example. Test the convergence of the series

$$\frac{x}{1} + \frac{1}{2}\cdot\frac{x^2}{3} + \frac{1.3.}{2.4}\frac{x^3}{5} + \frac{1.3.5}{2.4.6}\frac{x^4}{7} + \cdots$$

Soln

Here, $U_n = \dfrac{1.3.5.\cdots(2n-3)}{2.4.6.\cdots(2n-2)} \cdot \dfrac{x^n}{2n-1}$ — (1)

Replacing n by $n+1$, we get

$$U_{n+1} = \frac{1.3.5.\cdots(2n-3)(2n-1)}{2.4.6\cdots(2n-2)(2n)} \cdot \frac{x^{n+1}}{2n+1} \quad (2)$$

$$\therefore \frac{U_n}{U_{n+1}} = \frac{1.3.5.\cdots(2n-3)}{2.4.6\cdots(2n-2)}\frac{x^n}{2n-1} \times \frac{2.4.6\cdots(2n-2)2n}{1.3.5\cdots(2n-3)(2n-1)} \times \frac{2n+1}{x^{n+1}}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{2n}{2n-1} \times \frac{2n+1}{2n-1} \times \frac{1}{x}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{2n.(2n+1)}{(2n-1)^2} \times \frac{1}{x}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{\frac{2n}{2n}.\left(\frac{2n+1}{2n}\right)}{\left(\frac{2n-1}{2n}\right)^2} \times \frac{1}{x}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} = \frac{1+\frac{1}{2n}}{\left(1-\frac{1}{2n}\right)^2} \times \frac{1}{x}$$

Take $\underset{n\to\infty}{Lt}$ both sides, we get

$$\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = \underset{n\to\infty}{Lt} \left[\frac{1+\frac{1}{2n}}{\left(1-\frac{1}{2n}\right)^2}\right] \times \frac{1}{x} = \frac{1}{x}$$

Hence, by Rabo' test, the series is

(a) convergent if $\frac{1}{x} > 1$ i.e. $x < 1$.

(b) divergent if $\frac{1}{x} < 1$ i.e. $x > 1$.

But if $x = 1$ then $\underset{n\to\infty}{Lt} \frac{U_n}{U_{n+1}} = 1$

then this test fails.

When $x = 1$

$$\therefore \frac{U_n}{U_{n+1}} = \frac{2n(2n+1)}{(2n-1)^2}$$

$$\Rightarrow \frac{U_n}{U_{n+1}} - 1 = \frac{2n(2n+1)}{(2n-1)^2} - 1$$

$$\Rightarrow \frac{U_n}{U_{n+1}} - 1 = \frac{2n(2n+1) - (2n-1)^2}{(2n-1)^2}$$

$$= \frac{\cancel{4n^2} + 2n - (\cancel{4n^2} - 4n + 1)}{(2n-1)^2} = \frac{6n-1}{(2n-1)^2}$$

$$\Rightarrow n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{6n^2 - n}{(2n-1)^2} = \frac{\frac{6n^2 - n}{n^2}}{\left(\frac{2n-1}{n}\right)^2}$$

$$\Rightarrow n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{6 - \frac{1}{n}}{\left(2 - \frac{1}{n}\right)^2}$$

Take $\underset{n \to \infty}{Lt}$ both sides, we get

$$\Rightarrow \underset{n \to \infty}{Lt}\, n\left[\frac{U_n}{U_{n+1}} - 1\right] = \underset{n \to \infty}{Lt}\left[\frac{6 - \frac{1}{n}}{\left(2 - \frac{1}{n}\right)^2}\right] = \frac{6}{2^2}$$

$$\Rightarrow \underset{n \to \infty}{Lt}\, n\left(\frac{U_n}{U_{n+1}} - 1\right) = \frac{6}{4} = \frac{3}{2} > 1.$$

$\Rightarrow$ The given series is convergent when $x = 1$.